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MACRO-SITUATION AND NUMERICAL KNOWLEDGE BUILDING: THE ROLE OF PUPILS' DIDACTIC MEMORY IN CLASSROOM INTERACTIONS

ABSTRACT. This paper is based on a long-term didactic engineering about division problems (only in a numerical setting) at primary school. Situations and students' work are analyzed by means of a double theoretical framework: the theory of situations and the theory of conceptual fields (Vergnaud 1991). The analysis focuses mainly on classroom interactions and on the didactic memory from both the teacher perspective and the learner perspective: in particular, it not only investigates how didactic memory is managed by the teacher, but also how students recall past events or reread those events in a-didactic situations.

KEY WORDS: concepts-in-action, didactical engineering, didactical memory, division problems, operational invariant, numerical knowledge, theory of conceptual fields, theory of didactical situations, schemes

1. INTRODUCTION

The concept of the *teacher's* didactic memory was first proposed in Brousseau and Centeno's work in the early 1990s, in relation to the theory of didactic situations. More recently, the concept was reconsidered in terms of the anthropological theory of didactics (Matheron, 2001). The concept of a *pupil's didactic memory* will be studied here in the dual framework of Brousseau's (1997) *theory of didactic situations* and Vergnaud's (1996) *theory of conceptual fields*. The idea will be to present the research that led to the definition and development of this concept. In line with Brun and Conne's (1991) work in Geneva, an initial study was conducted (Flückiger, 2000) to identify this pupil-initiated memory phenomenon. Then, as part of a project by the Franco-Genevese research team on comparative didactics, the data was reanalyzed to determine how teacher's actions can elicit this memory (Flückiger and Mercier, 2002). After a description of the main results of the initial study, and a demonstration of the emergence of didactic memory in the pupil, a functional structure of how numerical knowledge is built will be proposed.

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2. The initial study

The initial study dealt with the emergence of numerical knowledge in the classroom. At the theoretical level, the study was aimed at testing the relevance of articulating the two theories that supplied the framework for the present experimental research. The mathematical object under examination was written calculation algorithms, more specifically, how *problem divisions* are studied in a Genevese class of fifth graders (approximately age 10). The term "*problem divisions*" is used to refer to the fact that the divisions in question may be difficult for pupils who have not yet studied long division. This implies that finding the right answer will require approaching a genuine mathematical question. The term is also used to express the fact that these problems were not everyday *division problems* like the story problems commonly given in this grade. The pupils worked in a purely numerical context. Fifth graders already have knowledge of addition, subtraction, and multiplication algorithms, which they have been taught in school, and they also know about the equivalence between multiplication and division in simple cases like those found in multiplication tables (for example, 10 divided by 5 equals 2, because 2 times 5 equals 10).

Note that the goal of the study was not to lead the children to invent a division algorithm that would later be instituted in the classroom. Nor was it a question of testing a new teaching method for written division problems, currently learned in fifth grade. The goal was rather to devise a *research methodology* for studying the genesis of numerical knowledge over time, under didactically controlled conditions.

2.1. *Longitudinal study*

At the macro-engineering level, the idea was to create learning conditions, in which meaning could be controlled during the teaching of a division algorithm. The corpus of data that we analyzed was collected in a classroom in Geneva and included all classes over an entire school year where the concept of division was taught (about 50 sessions).

The methodology traditionally associated with the theory of didactic situations is called *didactic engineering*. In a didactic-engineering approach, unlike a "naturalistic" type of observation, empirical data are compared and related to theoretical models in an organized way. In the present case, the goal was to find out how numerical learning takes place while working towards the elaboration of an algorithm for long division. The aim here is to attach meaning to this learning in a setting organized for that purpose and based on a chosen theoretical framework, the theory of didactic situations. Artigue (1990, 1992) defined didactic engineering as follows:

Didactic engineering, seen as a research methodology, is, firstly, characterized by an experimental schema based on class[room] 'didactic sequences', by which we mean based on the design, the production, the observation, and the analysis of teaching sequences. Here classically two levels are distinguished, *micro-engineering* and *macro-engineering*, depending on the size of the didactic sequences involved in the research (Artigue, 1992, p. 44).

2.1.1. *Method of study*

The experiment was organized around weekly cycles in which the micro-engineering level corresponded to sessions held in the classroom, and the macro-engineering level – by virtue of its duration – corresponded to the general experimental device. The weekly cycles were composed of one or more teaching sessions, followed by a consultation session among the members of the team (researchers and teacher). This cycle was repeated throughout the school year. The regular link maintained between classroom experimentation and analysis sessions is shown in Figure 1.

The initial sessions were derived directly from Kamii's (1994) work, conducted in reference to Piaget's theory. These sessions, called "calculation" sessions, were based on pupils' inventiveness in the face of a new type of arithmetic problem (here, division), i.e., one for which no specific algorithm has been taught as yet. However, the pupils already knew, for example, that 12 divided by 6 equals 2 because 2 times 6 are 12. So new knowledge can be built on that already acquired, the equivalence between multiplication and division.

Inventiveness alone does not suffice to move forward in the learning process, and, in fact, only the teacher's actions (here, controlled by the study) and the manipulation of didactic variables enabled the pupils' procedures to evolve.

2.1.2. *Didactic variables*

Two types of didactic variables were applied in this study. These were the numeric variables for the problems of long division used in the experiment and the variables involved in the setting up of the classroom sessions. The list of classroom sessions is included in the annex.

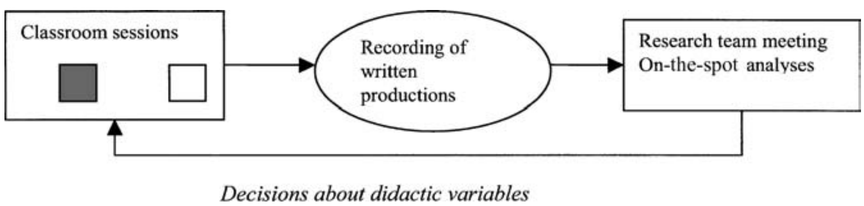


Figure 1. Weekly structure of the study.

Calculation sessions were organized temporally into “phases” (using the terminology proposed by Margolinas, 1993). During the *action* phases, pupils were given a division problem and had to work individually to find the answer. This was followed by a *communication* phase, during which the different answers (and procedures) found were presented to the class and compared. Based on the theory of didactic situations and a functional perspective on knowledge building, this is where the *formulation* and *validation* phases are articulated, with the conditions for moving from one to the other being among the questions raised in the study. For these sessions, the main didactic variable was, of course, the numerical variable. The researchers chose the numbers used in the sessions on the basis of previous research, which identified the difficulties connected with the numbers in long division and also according to the procedures elaborated by the pupils, which were analyzed at the end of each session.

For example, the first division problem proposed to pupils, $990 \div 9$, can be done “digit by digit”, or by first representing 990 as a sum of 900 and 90 and then dividing each component by 9. The second division problem selected for use “$1818 \div 9$” obliged the pupils to find new ways of doing long division.

Besides the calculation sessions other types of sessions, described briefly below, were set up in accordance with the *on-the-spot* analyses conducted each week.

Journal-writing sessions marked off the progression of the pupils’ work and the queries they raised. These sessions served as a support for the individual preparation of questions about the mathematical object “division”. In a personal mathematical diary used for this purpose only (and referred to as “Journal” in this text), the pupils had to write their answers to questions raised by the teacher. They knew this book would never be marked or checked. The questions would be of a temporal or epistemological nature. For example, “How are you getting along with division problem?” “What do you find most interesting about division problem?”

Some of the responses were selected and given to the whole class for later journal-writing sessions or for use as topics of *debate sessions*. The first debate session was based on a statement taken from a pupil’s journal: “My classmates’ results are different from mine because they use a different method from the one I do.” This idea was discussed first in small groups then by the whole class. It helped the pupils to consider the difference between method and result with respect to the uniqueness of the result of a calculation.

The debate-session setting was borrowed from Sensevy’s (1998) work on the study of fractions in elementary school. In his study, which focused on the temporal dimension of knowledge production in the classroom, Sensevy attempted to render the pupil's activity *chronogenic*, i.e., to have the pupils' productions move the learning process forward. This setting also creates conditions which make the pupil responsible for evoking past situations, thereby enabling the creation of connections between the private and classroom dimensions of memory. Also, after the fashion of what happens in a research community, *journal* writing can promote the cooperative dimension of scientific work in the classroom. It offers a medium for capturing the temporal dimension of knowledge production in a community, as stressed by Sensevy.

The methods tournament session was set up to compare the different procedures used in the classroom. It provided the opportunity for questioning the very notion of algorithm (efficiency, range of validity, etc.). In groups, the pupils demonstrated their methods of calculation to each other and debated them. Points were awarded for the speed, efficiency and variety of the methods each group put into play.

These different sessions supplied the variables that governed our study.

2.2. *A dual theoretical framework*

In the framework of a didactic system modeled by the teacher–pupil–knowledge triplet, the study focused on the elaboration of knowledge of division by the pupil subsystem, in a research-controlled didactic context. The question was, how do classroom interactions evolve in a situation where it is left up to the pupil to move the learning process forward and to discover new questions about the object under study.

In line with Brousseau's theory, the research methodology was engineered to create conditions that allow to trigger the dialectics necessary for a meaningful acquisition of the target knowledge. Brousseau modeled the different ways of functioning in terms of the *action*, *formulation*, and *validation* situations. When the pupil is interacting with the situation during the *action* phase, he/she is not necessarily capable of expressing the knowledge at play. This is achieved later in the course of the communication phases, where the pupil is led to *formulate* the knowledge and present it to others. At this point, the information must be understood and transformed by the interlocutor into a relevant *decision*.[1] In the theory, the *validation* situation represents the transition from empirical validation to an assertion recognized by all and integrated into known theorems. In Margolinas's (1993) terms, this involves creating conditions for moving from an *assertoric truth* to an *apodictic truth*,[2] which is brought about by scientific debate.

The idea in this study was to enrich the theory of didactic situations with the theory of conceptual fields. While the former served as a model for designing the experimental classroom setting, the latter was used to detect the operational invariants underlying the subjects' behavior. Vergnaud (1996) developed the theory of conceptual fields in an attempt to analyze the question of conceptualization, and the continuities and discontinuities that occur in the course of learning. Taking up Piaget's concept of *scheme* proposed in genetic epistemology, he centered his theory on the scheme–situation duality.

> The conceptual pair 'scheme–situation' is the keystone of cognitive psychology and of activity theory, for the simple reason that getting to know means adapting; it is the schemes that adapt, and they adapt to situations (Vergnaud, 2002; our translation).

Vergnaud defines a scheme as a fixed organization of activity for a particular class of situations.[3] A scheme is linked to the time course of the activity. The notion of *class of situations* is both innovative and essential in Vergnaud's theory, where a *conceptual field* is defined as a set of situations *and* concepts. A concept does not develop in isolation, but is part of an entire system of diverse concepts that develop jointly during the conceptualization of a notion. Hence, the notion of conceptual field.

> It is the situations that give meaning to the concepts, by way of the learner's activity; it is the concepts-in-action and the theorems-in-action contained in the schemes that enable these situations to be processed (Vergnaud, 2002; our translation).

Here, Vergnaud distinguishes two kinds of invariants that make the subject's action operational. *Concepts-in-action* permit the processing of information considered relevant to the situation at hand. By identifying these concepts, we can determine what the pupils have selected, in that situation, as appropriate information for processing the problem. *Theorems-in-action* are ones held to be true during action. While the concept of *scheme* is associated with that of *situation*, the notion of *operational invariant* is associated with *mathematical objects*, their properties, and their relations. In Vergnaud's theory, concepts-in-action and theorems-in-action are interpreted in reference to the concepts and theorems of mathematics.

The theory of conceptual fields, which is not a didactic theory *per se*, allows us to approach school learning in terms of its characteristic duration and nonlinearity. It allows us to see inside the pupil subsystem, the "black box" of the didactic system. It makes an essential contribution to understanding the cognitive facet of didactics. In the present analyses, the notion of scheme will be applied to the theory of didactic situations in order to analyze the pupils' productions.

Two levels of analysis are superimposed in our study. One reflects the level of didactic engineering in the context of the theory of didactic situations. There are two dimensions at this level. The macro-level deals with the long-term building of numerical knowledge about long division, and at the micro-level each session is divided into a phase of action followed by a phase of communication. The second level of analysis is that of the invariants in the context, with which the pupils had to deal in their calculations. The procedures invented by the pupils were then interpreted according to the theory of conceptual fields. This interpretation of the pupils' methods and their evolution allows the researchers to understand if the pupils' knowledge works well in registers anticipated by the theory of situations (action, formulation, and validation). The pertinency of the observations registered during this double evaluation allows the researchers to validate their theory.

Although we are unable to make a detailed list of every contribution to this research project, we shall indicate some of the most significant results concerning the object of division problem. This will be followed by further consideration of the pupils' didactic memory.

2.3. *Division as an object of learning*

Due to its complexity and the important place occupied in mathematical learning, division is particularly interesting for anyone hoping to understand numerical knowledge acquisition in elementary school children. Paradoxically, the fact that this subject matter is "didactically old," so to speak, is an interesting point in itself. The topic of division algorithms in school has been addressed from a number of angles, so it is possible to draw from results accumulated in the research over the years. The available studies have been conducted in the framework of the theory of didactic situations (didactic engineering, list of conceptions of division, etc.) or the theory of conceptual fields (error lists, integration of a scheme into an error explanation system, etc.). The body of findings obtained from different theoretical spheres offers a starting point for interrelating the two models of interest to us here by providing an original unit of analysis in didactics, the scheme. This approach is in line with the considerations brought to the fore by Brun, who emphasized the need to relate the original models of didactics to "lower" level models like those of developmental psychology.[4]

In the 1990s, the Genevese research team on mathematics teaching, headed by Brun, worked specifically on the issue of written calculation algorithms (Brun et al., 1991). The first series of studies dealt with the analysis of pupils' errors in written calculations. They showed that Brown and Van Lehn's (Brown and Van Lehn; 1980, Van Lehn, 1988) Repair Theory cannot account for all subtraction errors. According to Brun and Conne, systematic or recurring errors made by pupils are traces of the gradual construction of an algorithmic scheme. In this dynamic view of errors, pupils adapt their knowledge in order to progress in their calculations.

In a talk on written calculation algorithms presented in Geneva in 1996,[5] Brun showed how previously learned algorithms may or may not act as schemes. The criterion he used for schemes was adaptability. He raised the question of whether learned, automatized algorithms can be "decontextualized" and unraveled to uncover their numerical meaning. Another indicator used by Brun is the ability to communicate new procedures. To what extent can new procedures be imparted to, understood, or even actually put to use by others? The author's observations demonstrate the relevance of his scheme-based analysis of written calculation algorithms, and suggest that a previously learned algorithm can emerge in the form of a scheme that becomes available when a new situation is being processed.

This perspective was applied here by combining it with a more specifically didactic dimension, that of the conditions in which learning is taking place – i.e., the situations and how they are handled – and which trigger such adaptation.

3. Some results

3.1. *"Problematizing" division as an object of learning*

In this experiment, pupils were presented with a division problem on the blackboard, without ever having been taught any kind of algorithm to solve it. They had to think up ways to perform the calculation. How the pupils grasped the mathematical object of "division" and – far beyond the question of the answer obtained from the string of calculations they proposed – how they "problematized" their search for a solution, are interesting findings in themselves. In particular, the queries that emerged here about the existence or the uniqueness of a quotient are rare at this grade level! It is not just the numerical variable and the sequence of calculation sessions that enabled this to happen. As stated above – and this is a clear-cut result – the inventiveness of pupils does not alone suffice to move forward in the learning process. The *control variables* that dictated the experimental situation and determined the nature of the sessions proposed (journal, debates, etc.) allowed each pupil to share with the teacher the responsibility of advancing the learning process and undertaking the problematization of the question posed to the class as a whole. Along with the teacher's management of the dual didactic and research contract, the manipulation of the didactic variables is a condition for the emergence of true mathematical inquiry.

3.2. *Results of numerical knowledge building*

A scheme-based analysis of the data turned out not only to be compatible with the theory of didactic situations, the foundation of the engineering process, but also very fruitful from the standpoint of the results obtained. First, the results pertaining to numerical knowledge construction and to the theory of didactic situations itself will be presented. Then the issue of didactic memory will be addressed.

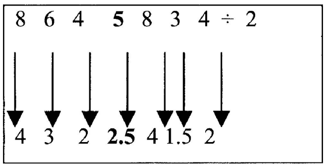
3.2.1. *The question of the remainder*

The analysis pointed to the invariants in the schemes that became operational during the pupils' activity. The detection of both the concepts considered relevant by the pupils and the theorems they used to treat the problem, served to pinpoint *topogenetic* shifts. This concept (extended in particular by Chevallard, 1985/1991) refers to the respective positions of the teacher and the pupil in their relationships to different objects. This dimension must be considered in conjunction with the *chronogenetic* dimension (Chevallard, 1995/1991), which accounts for temporal changes. The topics of classroom debates are indicative of how the pupil subsystem is evolving relative to the concept under study. For a given pupil, the operationalized invariants are a reflection of how his/her knowledge network is evolving in the conceptual field of, in this particular case, division.

For example, the question of the remainder gradually took over in the classroom debates. This question was first brought up by a newcomer in the class who had already studied the traditional division algorithm. He declared that the answer to "6 divided by 5" was "1 remainder 1". At the time, the class was unable to decide between the two answers proposed: "1 remainder 1" and "1 point 2". To conclude the session, a summary was made stating the disagreement and the two answers given, one supported by multiplication ($1.2 \times 5 = 6$) and the other by the Euclidean equation representing the division ($5 \times 1 + 1 = 6$). This became the topic of some highly interesting and heated debates in subsequent sessions.

The existence, relevance, and magnitude of the remainder with respect to the divisor are mathematical questions that supply the grounds for differentiating between the integer quotient and the decimal quotient. Associating a single quotient number to the dividend–divisor pair, or associating the quotient–remainder pair, is a choice which, in the absence of a concrete context, raises touchy questions like the uniqueness of the quotient–remainder pair, the nature of the numbers studied (integers or decimals), etc. These questions ended up leading the class to say to the teacher, "You have to tell us what set of numbers we're working in, N or R." Remember that this is grade 5 of elementary school!

Regarding the question of the division algorithm, approaching the data in terms of schemes pointed out the following: each partial quotient in the division algorithm is a Euclidean division, i.e., the entire quantity is not being divided. This was totally new compared to previously studied algorithms. Take the following production:



In the digit-by-digit processing done by this pupil, each successive number is divided in its entirety. By introducing several decimal points in the written quotient, the pupil failed to abide by the place notation system.

This example is reminiscent of Brun's observations regarding the transfer of previously learned schemes to a new algorithm.

In addition, subtraction, and multiplication problems, quantities are processed in their entirety (added, subtracted, or multiplied). In the case of the above division, the intermediate division "5 divided by 2" should produce the pair (2, 1), the remainder "1" being carried over to the next row to produce 18, which is then divided by 2. This is not what the pupil does; he produces the quotient 2.5. Providing a pair (quotient and remainder) as a result of division is indeed a completely new step in the calculation. The heated and recurrent debates between the pupils about the relevance of "leaving a remainder" illustrate the underlying mathematical difficulty of the division process, which in that case is perceived as uncompleted. This specificity of the division algorithm needs to be brought to the fore, particularly in teacher training programs. It is a difficulty that must be handled didactically, just like the virtual absence of subtraction in the procedures noted here. Although repeated subtractions form the basis for teaching the traditional division algorithm, the present experiment showed that the concept of subtraction may not be present in the conceptual field of division at the onset, and therefore has to be fully constructed, including for some teachers in initial training.

3.2.2. *Two major classes of procedures*

The systematic detection of invariants pointed to two major classes of procedures that contributed to the emergence of this numerical knowledge. Whether at the individual level or during interaction in the classroom, a key element lies in the link between the *calculation procedure* and the quotient *verification/invalidation* process. To understand the difficulty inherent in the nesting of the calculation and verification procedures, which are intertwined in division problem, the pupil's activity can be seen as referring to two different classes of problems (from the standpoint of the activated schemes).

The first, which concerns the search for the quotient, activates "calculation" schemes (in the strict sense of the term). The associated invariants enable the pupil to produce one or more numbers that belong to the field of possible answers for the numbers given (based on the way in which the division is introduced in this study, resolving $a \div b$ is equivalent for the pupils to finding the number q such as $b \times q = a$).

For instance, for the division of 2546 by 2, different approaches were observed. In one approach, the pupil transformed the numbers (dividend, divisor) or the operation. For example:

– 2546 became $2000 + 500 + 40 + 6$, then each term was divided,
– 2546 became a sequence of 2, 5, 4, 6, then each number was divided,
– 2546 divided by 2 became a multiplication with a "gap" e.g. $2 \times ???? = 2546$.

In all cases the aim was, *in fine*, to find a number which would be the result of the calculation.

In another approach, the pupils were concerned with the truth value. In this case, the result – partial or final – was assigned the value "right" or "wrong" with respect to the givens (initial numbers, operation to perform).

Here is an example of the second approach: the pupil is given the numbers to be divided, for example, 175 divided by 14. The list of the results found by the pupils is written on the board (here: 75; 63; 12.15; 12.5; 5.5) and each pupil can validate or invalidate the proposed quotients. For example, looking at the first quotient a pupil said "75 is too much, 5.5 is not enough." The teacher asked "Why?" The pupil replied "It's impossible, 10 times 75 already makes 750." By this theorem-in-action the pupil declares that the result of 75 is wrong.

The notion of scheme accounts for the general organization of the activity carried out in each of these situations, an activity which involves making the same pieces of mathematical knowledge work with various arrangements and variable levels of importance. It allows for constant reinterpretation of the situation, which is such that, whenever the givens change, the goals and subgoals are modified, along with the checking methods to use and the invariants that have become operational. The division scheme is based on the combination of two sub-schemes, one about the search for a result, the other about the verification/invalidation of this result. The selected invariants can be different, they may also be identical. It is the organization, the hierarchy of the procedures carried out in the activity which changes.

When observing the interactions in the classroom at a more macroscopic level, the procedure utilized for the division – and with it, the mathematical knowledge brought to bear – is assessed by peers verifying the correctness of a result made public. Inversely, a result thought to be the outcome of a valid procedure can be assessed either numerically (i.e., in terms of its nature, existence, or written notation) or as the solution to the division problem (Is this a quotient? Is it the correct quotient?). A genuine dialectical process that weighs the procedure against the result takes place here. Such dialectics seem to be a valid technique for building numerical rationality. Note also the necessity of having different procedures available, both for constructing the result and checking/invalidating it.

3.3. *Back to the theoretical models*

"Research modifies theory as much as theory determines experimentation" (Morf, 1972, p. 107; our translation). In the light of the results obtained here, let us now look at the model proposed by Brousseau in his theory of didactic situations. This theory, which served as a conceptual framework for setting up the teaching sequence tested here, offers a structure based on situations: action, formulation, validation, and finally, institutionalization.

3.3.1. *Verbalization phases*

In adidactical situations called action situations, a pupil's knowledge allows him/her to act upon the problem situation, which in return must provide the occasion either for modifying that action or for supporting a decision to act. The pupil's task is not to express, explain, justify, or even identify the knowledge at stake. By contrast, formulation situations are ones with a communicative goal. According to Brousseau, an original message that will be addressed to others (a single pupil or a group) is constructed from a known repertoire, but this dimension does not suffice for defining how knowledge functions. The key dimension is decision making. The knowledge formulated by the pupil must be converted into a relevant decision by the addressee who receives the information. To understand the adidactical calculation situations found in this study, it seemed useful to define what will be called the *verbalization phase*. The verbalization phase initiates the exchange phase, which is when communication takes place. Brun stressed the importance of the didactic organization of verbalization during the *didactic exchange* (Brun, 1994). As they verbalize, pupils talk about their own calculation procedure; they make the sequences of actions they performed known to others. This does not, however, make it a situation of formulation. From the standpoint of the theory, the verbalization phase is an action situation (see Figure 2).

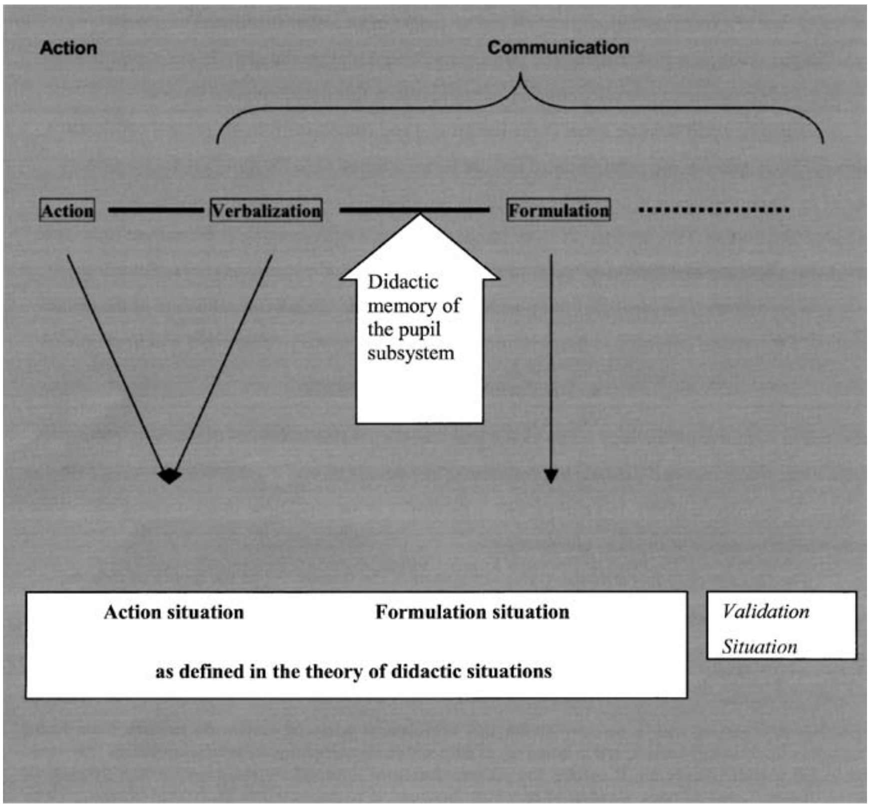


Figure 2.

Figure 2 depicts the locus of the verbalization phase, as it is understood in Brousseau's theory. The diagram also shows the point where the pupil's didactic memory is brought to bear in modifying the functioning of knowledge, a topic which will be addressed in the next section.

3.3.2. *From verbalization to formulation*

When pupils verbalize their procedure during a collective work session, they state the sequence of actions carried out to reach the goal of coming up with a numerical result. In this case, even if they can foresee some of their peers' reactions, they are not in a true position of expectation. They abide by the usual didactic contract in the classroom, where the pupil makes public some of his/her private actions. From the communicative standpoint, there is no real contract of collaboration, nor of opposition. It is the input provided in return by another pupil, based on the data given by the verbalizing pupil, that modifies the situation and grounds the exchange on the knowledge at play. Two critical positions can be found: either the pupil affirms the contradiction, or questions his/her own approach in the light of what was just said. In both cases, the prior situation must be reinterpreted to eliminate the discrepancy. During the reinterpretation process, which brings out the contradiction, a *decision* is made. Either the position evoked is maintained, or it is modified or even rejected altogether. In this case, the decision making occurs in a communication situation and no longer in an action situation. The action-verbalization situation is thus transformed into a genuine formulation situation. In our experimental system based on debates, contradictions play a central role in this transformation process.

Bringing a past didactic event to the foreground and comparing it to the present calls upon memory. Such episodes are pupil-initiated and are hence referred to as the *pupil's didactic memory*. One of its functionalities is to create the conditions for moving from an action situation to a formulation situation. This process involves comparing knowledge emerging at the current time with knowledge about a past situation (or point of view).

Now that we have identified this phenomenon relating to the adidactical nature of the macro-situation, a theoretical approach to the structure of the *didactic memory of the pupil* will be proposed.

4. The pupil's didactic memory

Research in cognitive psychology offers us the basic functional characteristics of memory: recursive functioning, which makes it possible to postpone certain decisions, and the capability to anticipate. Because of its didactic nature, this study looks in particular at the conditions in the didactic system that permits the emergence of memory phenomena.

The concept of didactic memory was developed in the framework of the theory of didactic situations by Brousseau and Centeno. The question raised by these authors concerned how the didactic system handles the temporary knowledge of pupils (Brousseau and Centeno, 1991; Centeno, 1995), so they focused on the didactic memory of the teacher.

> Endowing an organism with a memory allows this organism to postpone certain decisions without losing information likely to influence it, and in doing so, to keep within its processing capacities, conditions that would tend to fall outside. Above all, it enables the reinterpretation of information, and consequently, through the nesting of transformation rules, all sorts of recursive functions. It is the inevitable instrument of anticipations. [...] Thus, locally, memory acts concurrently with adaptation since it permits its postponement or its avoidance.

In fact, in the middle term, memory promotes adaptation. Indeed, it appears itself to be the result of an adaptation to interactions in which the subject must survive, foresee, adapt, and learn (Brousseau and Centeno, 1991, p. 199; our translation).

These authors showed that the teacher's memory allows him/her to organize the change in status that knowledge undergoes. They also showed that when teaching takes place without the teacher's memory of prior situations, knowledge is connected both differently and to a lesser extent.

Perrin-Glorian (1993) defined the notion of *recall situation* as the "re-reading" of a situation treated in a former session. The re-reading process is carried out by a pupil when called for by the teacher. Our studies have shown that *pupils themselves can initiate re-reading of past events* in a context that authorizes a certain amount of adidactical functioning. Anticipations about relevant mathematical objects have also been noted. These phenomena are linked to the characteristics of the macro-situation set up, and to the teacher's management of it. The contract is unusual, since responsibility for memory processes is, in fact, devolved upon the pupils. Journal writing is particularly conducive to this recalling process. The teacher's actions can enable (or not) the individual queries evoked via recall to become shared by the class. If so, individual memories are integrated into the collective knowledge-building process. We therefore define the concept of a *pupil's didactic memory*. Indeed, this is a manifestation of true *didactic* memory because it is steered by the didactic situation; there is an intention to teach and a didactic contract specially set up for this study. Our research study highlighted different levels of recall. Using this as a basis, we now propose a structuring of a pupil's didactic memory before identifying the characteristics which make it an essential concept in the understanding of didactic phenomena.

4.1. *Structuring of didactic memory*

Three types of these memory manifestations (Recall 1, 2, 3) were identified here, each fulfilling a different function in the genesis of numerical knowledge.

- In R1, a result or an old calculation that has now become relevant is remembered.
- In R2, a past event that points to a contradiction is evoked.
- In R3, a new class of problems is created.

R1: Recalling a Result or an Old-But-Now-Relevant Calculation.

Memory recall serves to extend the repertoire of results applicable to problems. At the onset, the results repertoire available for division problems is the one found in usual schoolbooks, since division is introduced in this macro-situation as the inverse of multiplication. The repertoire is expanded by adding locally instituted results, which are then integrated as "subroutines" into new calculations. An example of this was noted here for "6 divided by 5", which came up several times with the numerical variables used.

> Division [205650 ÷ 5]
> Pupil LU is explaining his approach involving an additive breakdown of the dividend:
> $200000 \div 5 = 40000$; $5000 \div 5 = 1000$; $600 \div 5 = 120$
> Two pupils spoke up when he came to $600 \div 5$
> *ME: I just remembered that 6 divided by 5 is 1.2*
> *ER: So did I. I remembered that/since you also have to have about a hundred, that makes 120*

The connection made between "6 divided by 5" and the associated result 1.2 (highly problematic at this point in the elaboration process) was thus handled as a whole, as an operational invariant in new situations that could be combined with other mathematical knowledge, in particular, knowledge about decimal numbers being learned at the time.

This type of manifestation which exhibits a link with previous knowledge is usual in a classroom. It is even often expected or incited by the teacher.

R2: Evoking a Past Event Revealing a Contradiction.

The journal is where the pupil's didactic memory emerges. In the example below, FR saw and put back into question what he had already noted as an inconsistency.

> During session 21, the division [826 ÷ 14] was proposed to the pupils. Among the answers debated, LU proposed 86.5, which he justified by the following calculations (written on the blackboard in a column by the teacher):
> $10 \times 80 = 800$; $4 \times 6.5 = 26$, so $14 \times 86.5 = 826$
> This result was invalidated by performing the multiplication
> $14 \times 86.5 = 1211$

Several times, FR was surprised:

> (...)
> *LI: Already if you do/the answer he found, 86, you drop the decimal point and you multiply by ten, that already makes 860 so it has to be wrong*
> *FR:* ***But that could convince us that it's*** *...*
> (...)
> *Teacher: What do we say for now about LU's?*
> *Pupils: It's wrong*
> *LI: Because from the beginning alone, you can tell it's wrong*
> *FR:* ***Not from the beginning alone, huh // because the beginning of the operation, I would have bet it was right***
> *Teacher: That's what you say*
> *FR: Yes*
> *Pupil: But that makes 1211 so it's wrong*

The result was checked by multiplication, and this concluded the debate. The answer 86.5 was declared wrong, but twice, FR manifested his reluctance. He repeated this during the journal-writing session. Below is FR's verbatim reply to a question about what had been a source of surprise during earlier lessons.

> *Why doesn't the division down here work? LU did this division*
> *826 ÷ 14*
> *80 × 10 = 800*
> *6.5 × 4 = 26.0*
> *800 + 26 = 826*
> *86.5 × 14 = 1211.0* (all operations were written in column format)

This remark can be understood by looking at session 20. At that time, the pupils were working on [345 ÷ 23]. Several procedures proposed by pupils had been validated, two of which were presented by their authors as follows (noted here in line format):

$5 \times 23 = 115$ and $10 \times 23 = 230$ so the correct quotient is 15,
$20 \times 15 = 300$ and $3 \times 15 = 45$ so the correct quotient is 15.

These two procedures are based on the distributivity of multiplication with respect to addition, and this is what FR thought he recognized in LU's procedure when he tried to find the quotient of 826 divided by 14. For him, there was an obvious contradiction, and the journal enabled him to express his doubts. He was reworking a mathematical invariant that was essential for performing the division algorithm. Breaking the dividend and divisor down into additions to isolate simplified quotients is a common erroneous procedure (e.g., $346 \div 123$ becomes $300 \div 100 = 3$; $40 \div 20 = 2$; $6 \div 3 = 2$; result 322). Of course, the teacher then made the didactic decision about whether or not to pass on FR's query to the class as a whole.

Finding contradictions, which generates a twofold process of reinterpreting the past and anticipating the future, is a frequent phenomenon in the present teaching context. Such contradictions may be verbalized by the pupil him/herself, who formulates certain invariants and then puts them back to work. Or a peer may point out a contradiction between two decisions separated in time.

This type of memory manifestation is part of the chronogenetic dimension of knowledge construction. The specific didactic conditions, and particularly the journal-writing sessions allowed the public emergence of this type of questioning. This work of invariants, which is a central aspect of the conceptualization, remains usually private to the pupils and is not publicly visible in the classroom.

R3: Creating a New Class of Problems.

The creation by a pupil of a new class of problems is indicative of a significant topogenetic change in the organization of the conceptual field, in this case, division. This reorganization which structures in a new way the total number of classes of situation connected with the concept of division has an impact on the memory function. This new structuring of situations will lead to a different practical approach to division problem because types of situations and schemes are dialectically connected.

Let us look at the following example:

> Division [$147097 \div 7$]
> During the debate, a pupil named VIA proposed:
> VIA: *You have to put a decimal point and then nines to infinity, as LI did for a hundred divided by three*

This pupil saw that the current situation was analogous to one previously encountered in "100 divided by 3," a division proposed by LI in the journal. To this calculation situation, VIA associated the production of a quotient of the type "infinite decimal," and based his answer on this scheme. In Vergnaud's theory, the notion of scheme is closely tied to that of class of situations. In a given situation, if a pupil makes the connection between a certain problem and an identified class of situations, the associated schemes can be activated, even combined, for the processing at hand. The creation of such classes contributes to overall cognitive efficiency and is therefore a fundamental process. The lengthy duration of our engineered teaching plan is one of the conditions that promotes the detection of cognitive reorganizations like these. On the basis of a remembered event, a new problem can be related to a prior problem, sometimes publicly in the present teaching setting.

When this happens, we can see how a pupil's knowledge is reorganized and how this allows him/her to anticipate the results of an entire series of calculations from the same class. This type of knowledge reorganization is a topogenetic event that temporarily isolates a class of situations associated to a quotient-finding scheme. In the present case, the scheme did not become a routine because[7] it was shown to be incorrect in later calculations. This approach is similar to the one found in neuroscience, where memory research focuses on operational processes and memory is seen as a generator of new categories (Rosenfield, 1988).

This third category of pupil's didactic memory manifestation shows a reorganization of the conceptual field of the pupil. A previously constructed scheme is associated with a category of new questions. This phenomenon is rarely so clearly formulated as in the example above, and is essential in the construction and the structuring of the knowledge.

4.2. *Characteristics of the pupil's didactic memory*

Two characteristics of the pupil's didactic memory should be underlined:

- firstly, that it is developed in a classroom community,
- secondly, that it is guided by the teacher's teaching activity.

Clearly, the phenomenon of the so-called collective memory exists in other contexts, in the same way as the intention to teach pupils exists outside of the classroom. However, it is the interplay between the two above-defined points which is at the core of didactic memory.

Of course, didactic memory is built on the basis of personal memories but this aspect does not completely cover the area studied. The didactic memory of the pupils' sub-system is built collectively. During the sessions the pupils did not only recall personal memories of how their learning had evolved. They based their comments on remarks made by a peer. The pupils' recall revealed the similarities and differences and even their disagreements regarding their knowledge. These discussion sessions more often brought to light not the disagreements of position between two pupils but the change in position of a peer. A frequently heard comment was: "before he/she agreed and now he/she disagrees." The pupils insisted strongly that any change in position regarding the learning process should be justified by the authors. The way the teacher dealt with such contradictions in the research project was essential.

For the researcher to understand how didactic memory played its role in speeding up the process, she had to analyze the teacher's classroom management in two ways.

– The intention to teach pupils a part of a body of knowledge during their scholastic period. The concept of a didactic contract is there to explain what cements the teacher-pupil relationship around the acquisition of a body of knowledge. Everyone in the class knows the learning at stake goes beyond the single classroom session.
– The classroom management through which the teacher first organizes the succession of mathematical concepts to be acquired by the pupils over a given period (chronogenesis) and then defines the teacher's and pupils' respective positions in relation to this knowledge (topogenesis)

In the study under examination a deliberate choice to use an a-didactic form of classroom management was made, to allow pupils' contributions to occur more freely including manifestations of didactic memory. The term "a-didactic" indicates that the pupils were allowed to assume certain responsibilities usually assumed by the teacher. For example, the responsibility of saying what is true, or of indicating, which objects of knowledge are the most pertinent. In a macro a-didactic situation such as the one under consideration, the progress in didactic time is partially devolved. This leaves them free to back-track or go "fast-forward" in public. Both these processes are encouraged in their journal writing by the teacher's questions.

Example: Journal no. 3

Question 1: What have you found out about how to do divisions?
Question 3: What would you like to know now about division?

In ordinary classroom organization, the teacher is the one who reminds the pupils of a certain lesson or exercise, or who says what they are going to learn next. In our study, the pupils are allowed to decide because of the special macro-research situation which was established. This places the pupils progressively in an unusual position regarding the acquisition of knowledge. More specifically by turning the class into a research group, the pupils are involved in a collaborative working environment in the sense that they have to built collectively what they have to learn.

Here the question at stake is to establish a written algorithm for performing division. However, the general path the pupils need to follow is marked out by the teacher due to his/her choice of didactic variables (numerical or instructional, for example, the instruction which turns a calculation session into journal writing). This "marking" became clear during classroom debates. The teacher's strategy involves the following steps:

– Marking out a field of investigation for the pupils. The teacher needs to check what the class has learnt previously (for example, how to do an addition). He/she then allows the class to discuss, which new mathematical objects they decide to work on.
– Highlighting any contradictions. The teacher encourages conflicting debate relative to the acquisition of the particular knowledge at stake (here, numerical knowledge involved in division).

Didactic memory is relative to this common object at stake and is not only the individual development of each pupil's memory. Regarding the time required by each pupil to complete a learning process, there is a process of acceleration connected to didactic memory. The collective examination of individual discoveries accelerates the learning process. Each pupil can be helped by the observations made by his/her peers.

5. Conclusion

In a didactic engineering approach based on the theory of didactic situations, the concept of scheme and the detection of invariants in the pupils' interactions proved useful in determining how knowledge evolves. Schemes and the accompanying analyses thus provide a relevant supplement to the theory of didactic situations. Concerning the conceptualization problems found in the pupil subsystem of the didactic system, using the scheme as a unit of analysis avoids the pitfall of taking a linear approach to knowledge building. The scheme turns out to be a good *didactic*-analysis instrument for taking into account the recursive and anticipatory functioning of didactic memory. By combining the theory of didactic situations and the theory of conceptual fields to analyze how knowledge evolves over time, it is possible to rethink the concept of didactic memory from the standpoint of the pupil subsystem. This concept cannot be dissociated from the knowledge-imparting process. Its role in transforming verbalization problems into true formulation situations is essential to the construction of numerical knowledge.

By relying on the notions of scheme and class of situations, the present study provided insight into how the conceptual field of division is elaborated under specific didactic conditions. Understanding this elaboration process requires considering the duration of school learning and didactic memory phenomena specific to the pupil subsystem. This study demonstrated the existence of the pupil's didactic memory, and showed how it is necessary in this type of didactic contract.

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NOTES

1. The theory of didactic situations was couched in terms of game theory, where *decision* is a key concept.
2. Mathematical propositions are *apodictic* and not *assertoric*. Here, the terms *assertoric* refers to Kant's modalities.
3. In the theory of conceptual fields, "class of situations" is understood as "class of problems."
4. On this subject, see Brun (1994).
5. Piaget-Vygotsky Congress, Geneva, 1996.
6. Though this is unusual in Genevese elementary schools, the teacher introduced the symbols N and R to refer to the natural and real numbers.
7. Term borrowed from Saada-Robert and Brun (1996).

ANNEX: LIST OF CLASSROOM SESSIONS

	Divisions		
Session (N)	n	Dividend	Divisor
1	1	990	9
	2	1818	9
2	3	2546	2
	4	2592	6
3	5	345	23
4	6	720	20
	7	426	2
5	8	425	5
	9	175	14
6	10	180	15
	11	427	7
7	12	4500	150
	13	633	3
8	14	8645834	2
9	15	35 787	3
10	J1	Journal no. 1	

(*Continued on next page*)

(*Continued*)

	Divisions		
Session (*N*)	*n*	Dividend	Divisor
11	16	8324782	2
	17	645357	3
12	18	405	45
	19	315	45
	20	3780	45
13	21	2000	16
14	J2	Journal no. 2	
15	22	3618	18
16	Debate session		
17	23	4824266	2
	24	4962	2
	25	645354	6
18	Posting results		
	26	525575	5
	27	124892	4
	28	35791	2
19	29	8324782	2
	30	55155	5
	31	12143	1
20	32	816328	8
	33	345	23
	34	71	71
21	35	826	14
	36	6	5
22	37	205650	5
	38	465	155
23	39	17	5
	40	4854816	4
24	41	14 497	7
	42	301 725	5
25	J3	Journal no. 3	
26	43	304515	5
	44	369	246
27	45	1268148	4
	46	1545005	5
	47	180	12

(*Continued on next page*)

(*Continued*)

	Divisions		
Session (N)	n	Dividend	Divisor
28	48	100	3
29	List of divisions		
to	List of methods		
32			
33	49	65724	2
	50	3780	90
	51	3612	12
34à36	Tournament of methods		
	52	147097	7
37	J4	Journal no. 4	
	53	25	2
38	54	81822	3
	55	995	9
39	Debate regarding the remainder		
40			
41	56	6412	8
42	57	3615	12
	58	224	7
43	59	167	8
44	60	75035	25
45	61	266	4
46	62	2186	5
	63	801	20
	64	375	12
	65	368	7

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